SOME LAWS OF SPACECRAFT ORBIT CONTROL

PMM Vol. 32, No. 3, 1968, pp. 513-516

D.A. MAMATKAZIN
(Moscow)

(Received July 31, 1967)

We consider the motion of a spacecraft acted on by a controlling acceleration in the plane perpendicular to the absolute velocity vector of its center of gravity. Control laws which make it possible to obtain the solution in analytic form are developed.

1. The motion of a spacecraft in a central gravitational field under the action of the controlling acceleration **W** in the plane perpendicular to the absolute velocity vector **V** of its center of gravity O_1 can be conveniently considered in a rotating right-hand orthogonal system $Oxyz$ whose axis y coincides with the radius vector **r** constructed from the center of gravity O to the point O_1, and whose axis x is directed in the direction of motion in such a way that **V** lies in the plane xy. The orientation of the axes $Oxyz$ relative to the inertial coordinates $O\xi\eta\zeta$ is defined (Fig. 1) by the longitudinal Ω of the ascending node, the

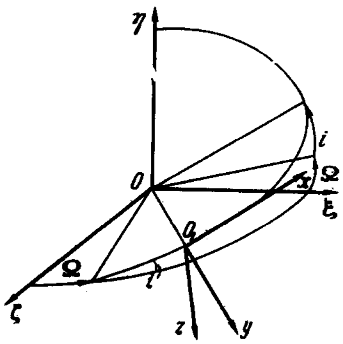

Fig. 1

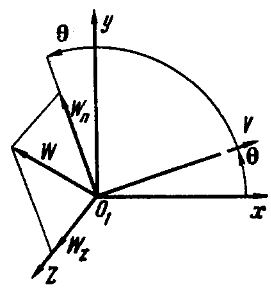

Fig. 2

inclination i of the instantaneous orbital plane relative to the equator, and the range angle u.

The controlling acceleration vector **W** is mapped onto the normal and binormal to the trajectory of motion by the projections W_n and W_z; the inclination of the absolute velocity vector **V** to the local horizon is defined by the angle θ (Fig. 2).

The equations of motion of the spacecraft are

$$V_x^{\cdot} = -W_n \sin\theta + \omega_z V_y, \quad V_y^{\cdot} = W_n \cos\theta - \omega_z V_x - g$$
$$0 = W_z + \omega_y V_x, \quad \omega_z = -V_x/r, \quad g = g_0 (R_0/r)^2 \qquad (1.1)$$

The rates of change of the angles of the rotating axes relative to the inertial axes are defined by the familiar differential equations

$$\frac{d\Omega}{dt} = \omega_y \frac{\sin u}{\sin i}, \quad \frac{di}{dt} = \omega_y \cos u, \quad \frac{du}{dt} = -\omega_z - \omega_y \sin u \operatorname{ctg} i \qquad (1.2)$$

The exact solution of this problem for $W_n = 0$ and W_z = const for a circular orbit is cited in [1 and 2], while [3] also contains an approximate solution for the case of an elliptical orbit. Paper [4] contains a solution for the case $W_n = 0$, $W_z = kr^{-3}$. Further on we shall write out the laws of variation of W_n and W_n which enable us to obtain the exact solution of the problem of motion of the spacecraft in general form (for circular, elliptical, parabolic, and hyperbolic orbits).

2. Equations (1.1) have the energy integral

$$^1/_2V^2 - gr = h \tag{2.1}$$

Moreover, the first of these equations with allowance for the fourth can be transformed into

$$\frac{d\,(V_x r)}{dt} = -W_n \frac{rr^{\cdot}}{V} \tag{2.2}$$

This equation can be integrated in quadratures with the aid of Eq. (2.1) if the projection of the controlling acceleration onto the normal to the trajectory of spacecraft motion is $W_n = W_n\,(r)$. Kinematic Eqs. (1.2) are integrable if the projection of the controlling acceleration onto the binormal to the trajectory varies according to the law

$$W_z = KV_x^2/r \qquad (K = \text{const}) \tag{2.3}$$

In this case, by virtue of the third and fourth equations of (1.1), we have

$$\omega_y = K\omega_z \tag{2.4}$$

and the last two equations of (1.2) are reducible to an equation in total differentials. On integrating the latter, we obtain

$$\cos i - K \sin u \sin i = k, \quad k = \cos i_0 - K \sin u_0 \sin i_0 \tag{2.5}$$

We also note that by virtue of the second equation of (1.2), the argument i reaches its extremal values i_* when

$$u = {}^1/_2\pi + m\pi \qquad (m = 1,\ 2,\ \ldots,\ n)$$

Eqs. (2.5) then become

$$-(1 + K^2)x_*^2 + 2kx_* + K^2 - k^2 = 0 \qquad (x_* = \cos i_*) \tag{2.6}$$

With allowance for (2.4) and (2.5) we can reduce the second equation of (1.2) to

$$\frac{dx}{\pm\sqrt{-(1+K)^2x^2 + 2kx + K^2 - k^2}} = -\,\omega_z dt \qquad (x = \cos i) \tag{2.7}$$

The quantity ω_z is a negative function of fixed sign, so that both sides of equation (2.7) are always positive.

Since the differential dx changes sign when the argument x passes through the extremal value $x = x_*$, and since the left side of (2.7) is always of fixed sign, we must break up the integration limits. To ensure positiveness of the left side of (2.7) in various portions of the orbit, we set

$$\frac{dx}{\pm\sqrt{-(1+K^2)\,x^2 + 2kx + K^2 - k^2}} = \frac{dx\,\text{sign}\,x^{\cdot}}{\sqrt{-(1+K^2)\,x^2 + 2kx + K^2 - k^2}}$$
$$\text{sign}\,x^{\cdot} = \text{sign}\,(K\cos u) \qquad (i > 0) \tag{2.8}$$

We can now write the integral of the left side of (2.7) as

$$J = \int\limits_{x_0}^{x_k} \frac{dx\,\text{sign}\,(K\cos u)}{\sqrt{-(1+K^2)\,x^2 + 2kx + K^2 - k^2}} \tag{2.9}$$

In the case of a circular orbit ω_z = const and the right side of (2.7) can be integrated in elementary fashion. If the orbit is not circular, we must first express the time in terms of the radius,

$$dt = \frac{dr}{V_y} = \frac{dr}{\sqrt{V^2 - V_x^2}} \operatorname{sign} r^{\cdot} \qquad (2.10)$$

Here V_x is a function of the radius r which we know once we have integrated (2.2). With allowance for (2.10) we can write the integral of the right side of (2.7) as

$$J = -\int_{r_0}^{r_k} \omega_z \frac{dr}{\sqrt{V^2 - V_x^2}} \operatorname{sign} r^{\cdot} \qquad (2.11)$$

This integral is easy to compute if the controlling acceleration normal to the trajectory of motion is given by the law

$$W_n = AV/r \qquad (A = \text{const}) \qquad (2.12)$$

In fact, on integrating (2.2) we find in this case that

$$V_x r = B - Ar, \qquad B = (V_{x0} + A) r_0 \qquad (2.13)$$

and (2.11) becomes

$$J = \int_{r_0}^{r_k} \frac{(B - Ar)\, dr \operatorname{sign} r^{\cdot}}{r\sqrt{ar^2 + br + c}} \qquad \begin{pmatrix} a = 2h - A^2,\ c = -B^2 \\ b = 2(g_0 R_0^2 + AB) \end{pmatrix}$$

From this expression we find that the extremal values of the radius r occur with $ar_*^2 + br_* + c = 0$.

Since the integrand in (2.11) is always of fixed sign, and since the differential dr changes sign at the instant when $r = r_*$, we must break up the limits of integration if r passes through an extremum. To simplify our discussion, we assume everywhere from now on that the arguments x and r do not pass through extremal values x_* and r_* throughout the entire time of action of the controlling accelerations W_n and W_z on the spacecraft. Here $\operatorname{sign} x^{\cdot} = \operatorname{sign} x_0^{\cdot}$ and $\operatorname{sign} r^{\cdot} = \operatorname{sign} r_0^{\cdot}$. Transforming the first two equations of (1.2) we obtain

$$\frac{d\Omega}{di} = \frac{\operatorname{tg} u}{\sin i}$$

With allowance for (2.5) and (2.8), we transform this equation into

$$\Omega - \Omega_0 = \int_{x_0}^{x_k} \frac{(x - k)\, dx \operatorname{sign}(K \cos u)}{(x^2 - 1)\sqrt{-(1 + K^2) x^2 + 2kx + K^2 - k^2}} \qquad (x = \cos i) \qquad (2.14)$$

Under the above assumption concerning the way in which the arguments x and r vary during the action of the controlling accelerations, integration of (2.9) yields

$$J = -\frac{\operatorname{sign}(\cos u_0)}{\sqrt{1 + K^2}} \left[\arcsin \frac{-(1 + K^2)\cos i + k}{\varkappa} - \arcsin \frac{-(1 + K^2)\cos i_0 + k}{\varkappa}\right]$$

$$(\varkappa = K\sqrt{1 + K^2 - k^2})$$

and integration of (2.11) gives us

$$J = \operatorname{sign} r_0^{\cdot} \left[\left(\arcsin \frac{br + 2c}{r\sqrt{-\Delta}} - \arcsin \frac{br_0 + 2c}{r_0\sqrt{-\Delta}}\right) + \frac{A}{\sqrt{-a}}\left(\arcsin \frac{2ar + b}{\sqrt{-\Delta}} - \arcsin \frac{2ar_0 + b}{\sqrt{-\Delta}}\right)\right] \qquad (a < 0,\ \Delta = 4ac - b^2 < 0)$$

$$J=\operatorname{sign} r_0^{\cdot}\left[\left(\arcsin\frac{br+2c}{r\sqrt{-\Delta}}-\arcsin\frac{br_0+2c}{r_0\sqrt{-\Delta}}\right)-\right.$$
$$\left.-\frac{A}{\sqrt{a}}\ln\frac{2\sqrt{a(ar^2+br+c)}+2ar+b}{2\sqrt{a(ar_0^2+br_0+c)}+2ar_0+b}\right]\qquad(a>0)$$
$$J=\operatorname{sign} r_0^{\cdot}\left[2\left(\operatorname{arctg}\frac{\sqrt{br+c}}{B}-\operatorname{arctg}\frac{\sqrt{br_0+c}}{B}\right)-\frac{2A}{b}(\sqrt{br+c}-\sqrt{br_0+c})\right]\quad(a=0)$$

In the case of a circular orbit, i.e. when $\Delta=0$, we have

$$J=\frac{V_0}{r_0}(t-t_0)\tag{2.15}$$

From (2.10) we have

$$t-t_0=\frac{\operatorname{sign} r_0^{\cdot}}{a}\left[\sqrt{ar^2+br+c}-\sqrt{ar_0^2+br_0+c}+\right.$$
$$\left.+\frac{b}{2\sqrt{-a}}\left(\arcsin\frac{2ar+b}{\sqrt{-\Delta}}-\arcsin\frac{2ar_0+b}{\sqrt{-\Delta}}\right)\right]\qquad(a<0,\ \Delta<0)$$
$$t-t_0=\frac{\operatorname{sign} r_0^{\cdot}}{a}\left[\sqrt{ar^2+br+c}-\sqrt{ar_0^2+br_0+c}-\right.$$
$$\left.-\frac{b}{2\sqrt{a}}\ln\frac{2\sqrt{a(ar^2+br+c)}+2ar+b}{2\sqrt{a(ar_0^2+br_0+c)}+2ar_0+b}\right]\qquad(a>0)$$
$$t-t_0=\frac{2}{3b^2}[(br-2c)\sqrt{br+c}-(br_0-2c)\sqrt{br_0+c}]\qquad(a=0)$$

and (2.14) yields

$$\Omega-\Omega_0=\frac{\operatorname{sign}(\cos u_0)}{2}\left\{\arcsin\frac{1}{\varkappa}\left[-\frac{(k+1)^2}{\cos i+1}+K^2+1+k\right]-\right.$$
$$-\arcsin\frac{1}{\varkappa}\left[-\frac{(k+1)^2}{\cos i_0+1}+K^2+1+k\right]+$$
$$\left.+\arcsin\frac{1}{\varkappa}\left[-\frac{(k-1)^2}{\cos i-1}-K^2-1+k\right]-\arcsin\frac{1}{\varkappa}\left[-\frac{(k-1)^2}{\cos i_0-1}-K^2-1+k\right]\right\}$$

In the special case where $K=0$, Eqs. (1.2) and (2.4) yeild

$$\Omega=\Omega_0,\qquad i=i_0,\qquad u-u_0=-\int_{t_0}^{t}\omega_z d\tau\tag{2.16}$$

Recalling (2.10) and the last equation of (2.16), we find from (2.11) that

$$J=u-u_0$$

All of the possible orbits described by Formulas (2.15) are conic sections in the scanning plane. The orientation of these sections varies with time,

the orbit is elliptical for $a<0$, $\Delta<0$
the orbit is circular for $a<0$, $\Delta=0$
the orbit is parabolic for $a=0$
the orbit is hyperbolic for $a>0$.

The motion of the orbital plane is characterized by the last formula of (2.15); its character does not depend on the shape of the orbit. The set of Formulas (2.1), (2.5), (2.13) and (2.15) enables us to predict the future parameters of the orbit and the position of the spacecraft under action by the controlling acceleration W with the components $W_y=AV/r$ and $W_z=KV_x^2/r$, and thus to effect the required control in the class of orbits with a constant energy integral.

3. For example, let us determine the transfer factors K and A for control laws (2.3)

and (2.12) which ensure tranfer of the spacecraft from an elliptical orbit to a precalculated circular orbit. From the angles Ω_0 and i_0 at rhe beginning of control and the angles Ω_k and i_k at the end of control we can determine the value of K which ensures coincidence of the plane of the spacecraft orbit with the required plane by solving the last equation of (2.15). According to the elliptical theory of motion, the radius of a circular orbit with a prescribed energy integral is given by Formula

$$r_k = -g_0 R_0^2 / 2h$$

the velocity on the orbit is

$$V_{xk} = V_k = \sqrt{-2h}$$

Having determined r_k and V_k, we can find the transfer factor A in accordance with (2.13) from Formula

$$A = \frac{V_{xk} r_k - V_{x0} r_0}{r_0 - r_k}$$

On attainment of the equalities $\Omega = \Omega_k$ and $i = i_k$ the controlling acceleration W_z is terminated and the orientation of the orbital plane is recorded. On attainment of the equalities $r = r_k$ and $V_x = V_{xk}$ the controlling acceleration W_n is terminated, and the shape of the orbit is recorded (according to (2.13), the area integral becomes constant).

The method of determining the factor K remains similar to the above in the other possible cases of controlled motion. The factor A is determined from the prescribed value of area integral (2.13) at the end of the controlled motion and from the orientation of the orbit of precalculated shape in space.

BIBLIOGRAPHY

1. Illarionov, V.F. and Shkadov, L.M., Rotation of the plane of a circular satellite orbit. *PMM* Vol. 26, No. 1, 1962.
2. Gus'kov, Iu.P., A method of orientating the plane of a circular orbit of a satellite. *PMM* Vol. 27, No. 3, 1963.
4. Lass, H. and Solloway, C.B., Motion of a satellite under the influence of constant normal thrust. ARS Journal Vol. 32, No. 1, 1962.
5. Kopnin, Iu.M., On the problem of turn the orbit plane of a satellite. Kosmich. Issled. Vol. 3, No. 4, 1965.

Translated by A.Y.